Robot Vision:
Projective Geometry

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SS 2025

Learning goals

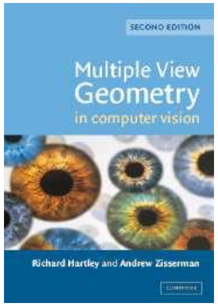
- Understand homogeneous coordinates
- Understand point, line, plane parameters and interpret them geometrically
- Understand point, line, plane interactions geometrically
- Analytical calculations with lines, points and planes
- Understand the difference between Euclidean and projective space
- Understand the properties of parallel lines and planes in projective space
- Understand the concept of the line and plane at infinity

Outline

- Differences between euclidean and projective geometry
- 2D projective geometry
  - 1D projective geometry
  - Homogeneous coordinates
  - Points, Lines
  - Duality
- 3D projective geometry
  - Points, Lines, Planes
  - Duality
  - Plane at infinity

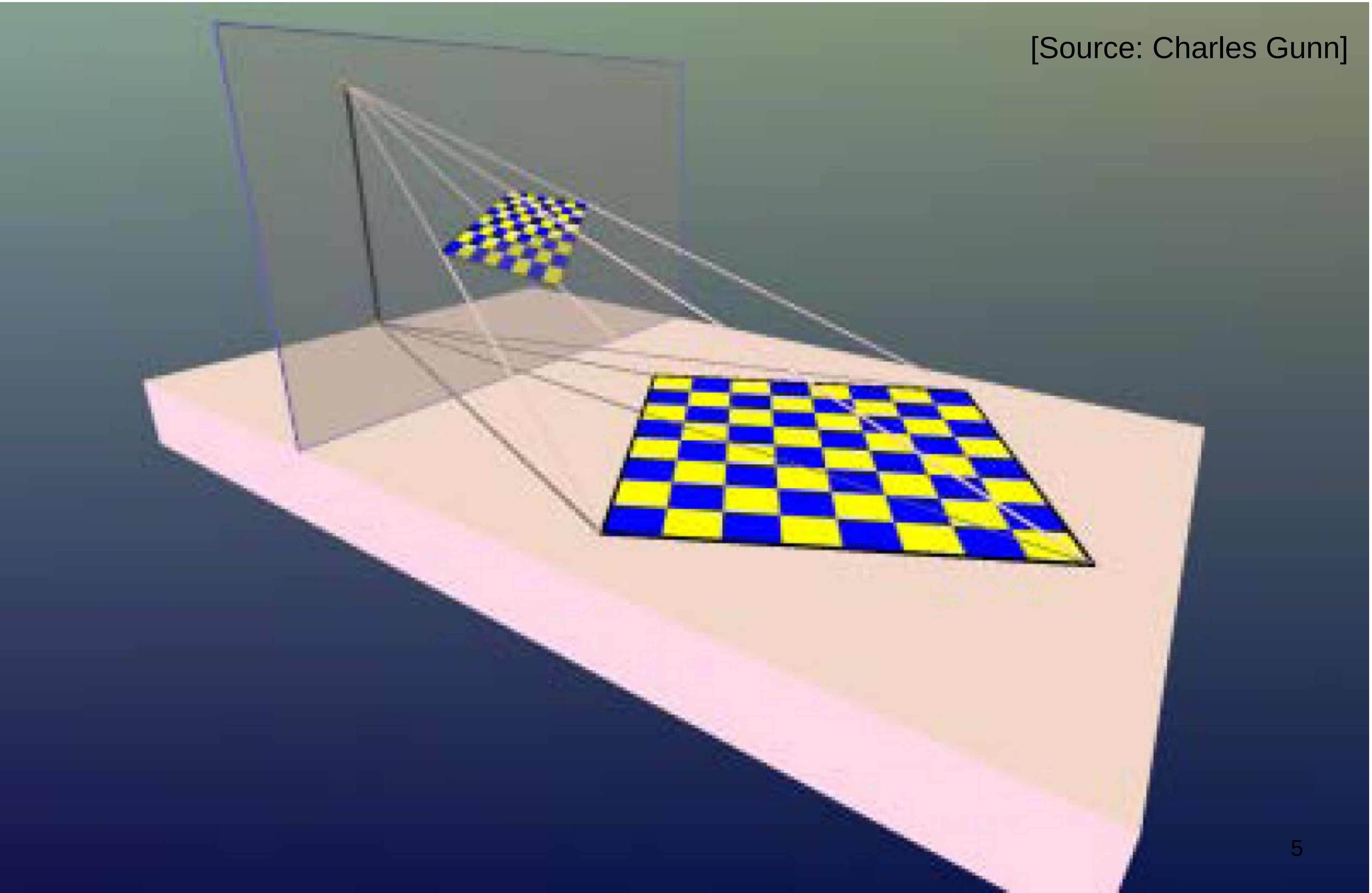
Literature

- Multiple View Geometry in Computer Vision. Richard Hartley and Andrew Zisserman. Cambridge University Press, March 2004.



- Mundy, J.L. and Zisserman, A., Geometric Invariance in Computer Vision, Appendix: Projective Geometry for Machine Vision, MIT Press, Cambridge, MA, 1992
- *Available online: www.cs.cmu.edu/~ph/869/papers/**zisser**-**mundy**.**pdf***

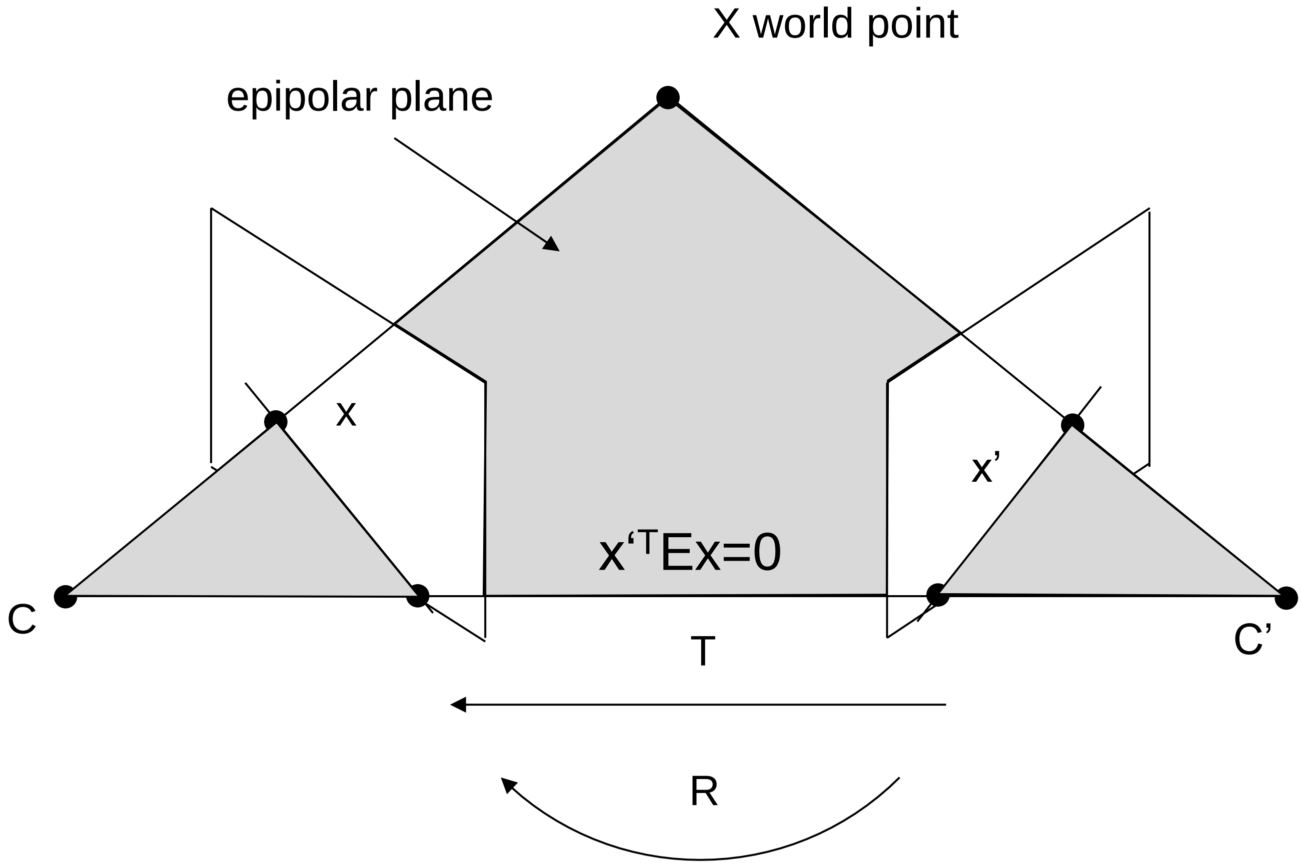
Motivation – Image formation



[Source: Charles Gunn]

Motivation – Parallel lines

Motivation – Epipolar constraint



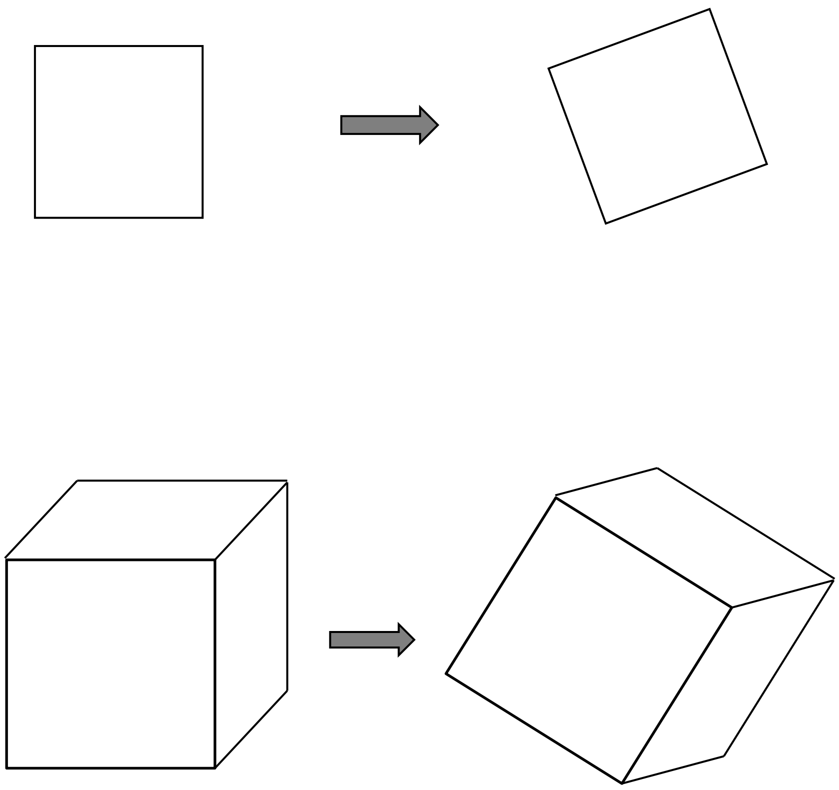
Euclidean geometry vs. projective geometry

Definitions:

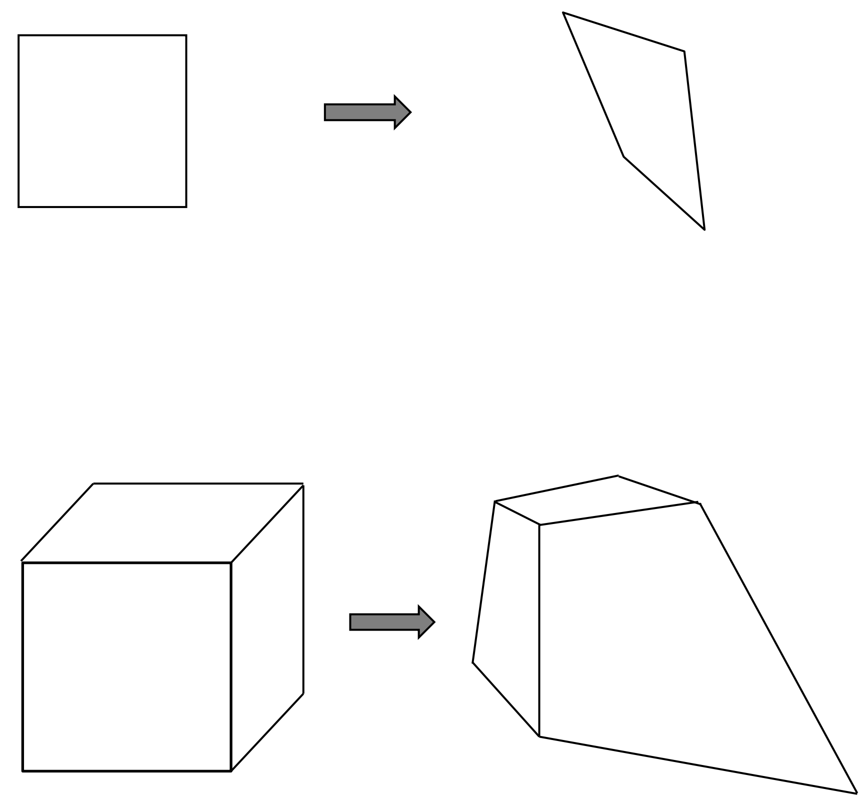
- **Geometry** is the teaching of points, lines, planes and their relationships and properties (angles)
- Geometries are defined based on **invariances** (what is changing if you transform a configuration of points, lines etc.)
- Geometric transformations of Euclidean geometry **preserve** distances
- Geometric transformations of projective geometry do **NOT preserve** distances

- Projective geometry was developed to explain the perspective changes of three-dimensional objects when projected to a plane.

Difference between Euclidean and projective transformation



Euclidean transformation



Projective transformation

Projective Geometry
2D Projective Geometry

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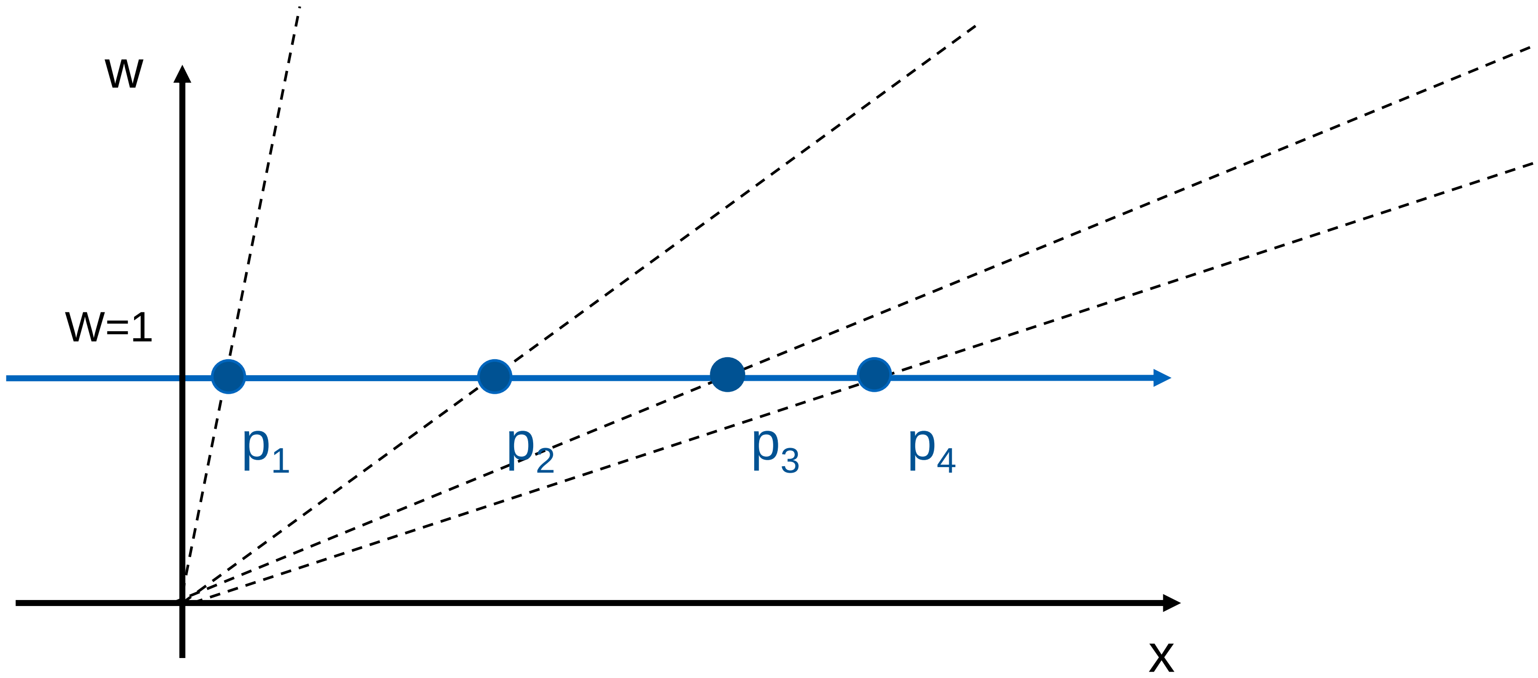
2D projective geometry

- Homogeneous coordinates
- Points, Lines
- Duality

1D Euclidean geometry

Euclidean coordinate:
$p_1=[x]$

1D projective geometry



homogeneous coordinate:
p_1=[x,w] ≈ [$\hat{x}$,1]

2D case - Homogeneous coordinates

- projective plane = Euclidean plane + a new line of points
- The projective space associated to R3 is called the projective plane P2.

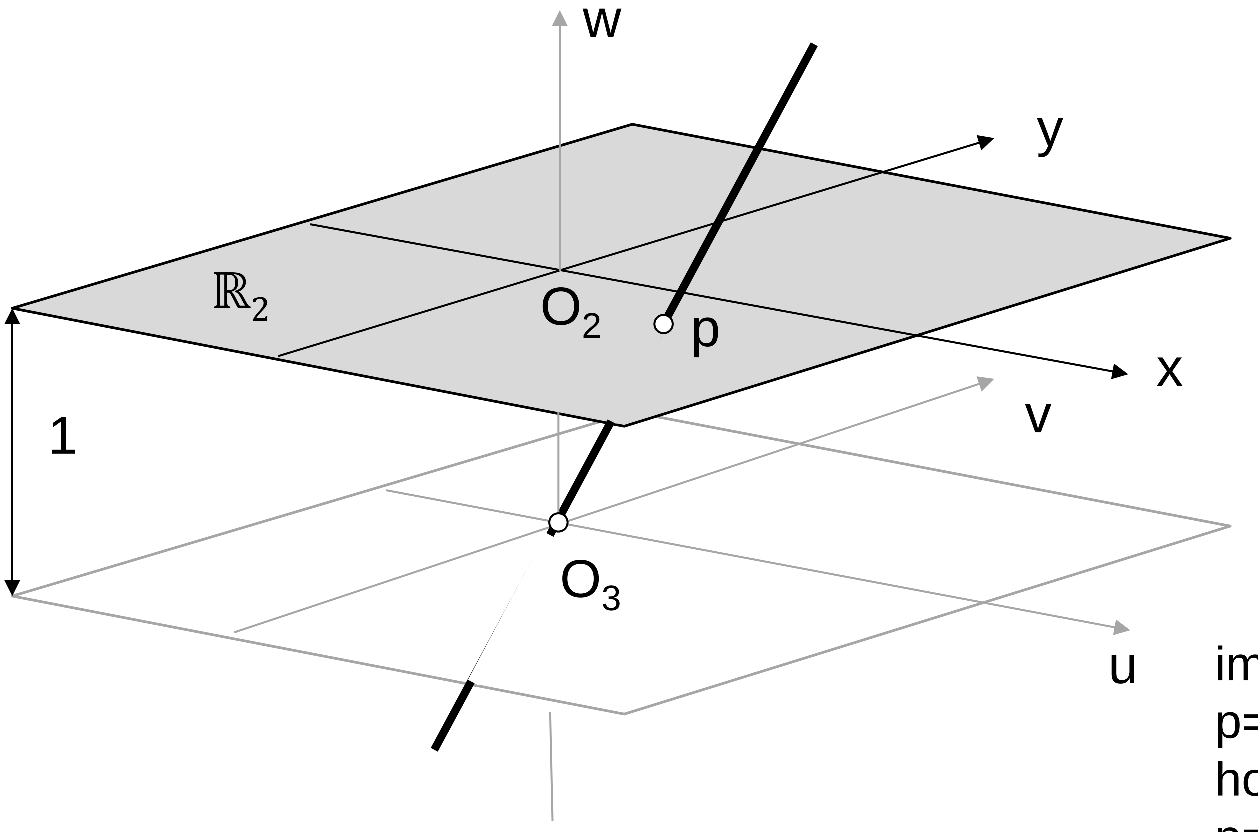
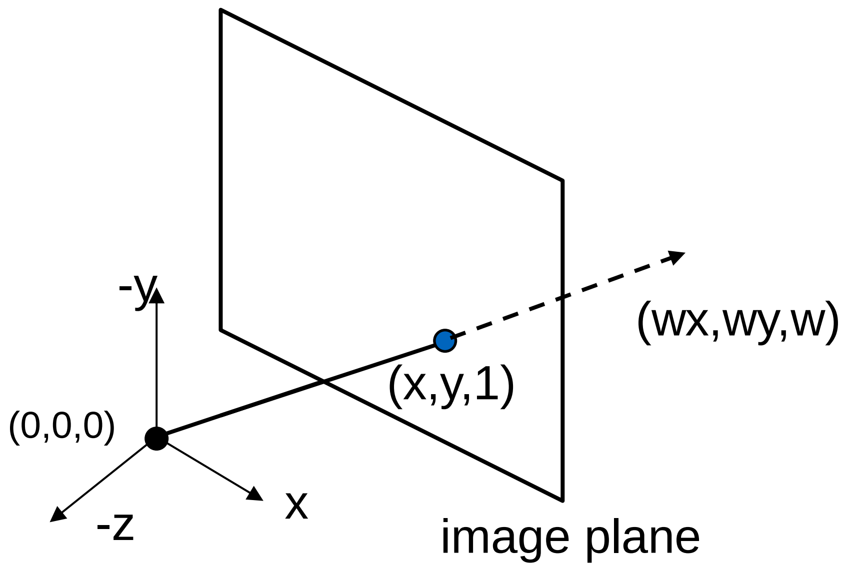


image coordinate:
p=[x,y]
homogeneous coordinate:
p=[u,v,w] ≈ [$\hat{u}$, $\hat{v}$ 1]

Points

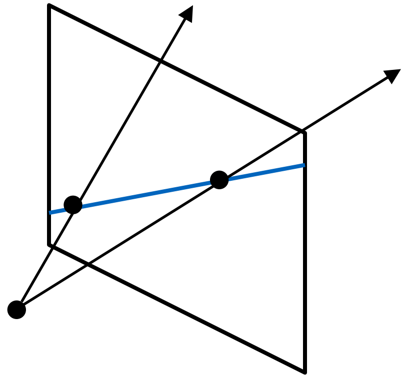
- A point in the image is a ray in projective space



- Each point (x,y) on the plane is represented by a ray (wx,wy,w)
  - all points on the ray are equivalent: (x, y, 1) $\cong$ (wx, wy, w)

Lines

- A line in the image plane is defined by the equation ax + by + cz = 0 in projective space
- [a,b,c] are the line parameters



- A point [x,y,1] lies on the line if the equation ax + by + cz = 0 is satisfied
- This can be written in vector notation with a dot product:

$$0 = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathbf{l}^{\mathbf{T}} \quad \mathbf{p}$$

- A line is also represented as a homogeneous 3-vector **l**

Calculations with lines and points

- Defining a line by two points

$$l = x \times y$$

- Intersection of two lines

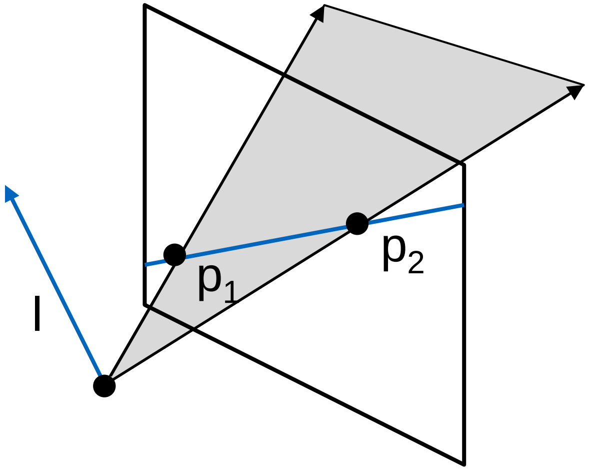
$$x = l \times m$$

- Proof:

$$l = x \times y$$
$$x^T l = y^T l = 0$$
$$x^T(x \times y) = y^T(x \times y) = 0 \text{ (scalar triple product)}$$

Geometric interpretation of line parameters [a,b,c]

- A line **l** is a homogeneous 3-vector, which is a ray in projective space
- It is $\perp$ to every point (ray) **p** on the line: $\mathbf{l}^T \mathbf{p}=0$



What is the line **l** spanned by rays $\mathbf{p_1}$ and $\mathbf{p_2}$?

- **l** is $\perp$ to $\mathbf{p_1}$ and $\mathbf{p_2}$ $\Rightarrow$ $\mathbf{l} = \mathbf{p_1} \times \mathbf{p_2}$
- **l** is the plane normal

Point and line duality

Duality principle:

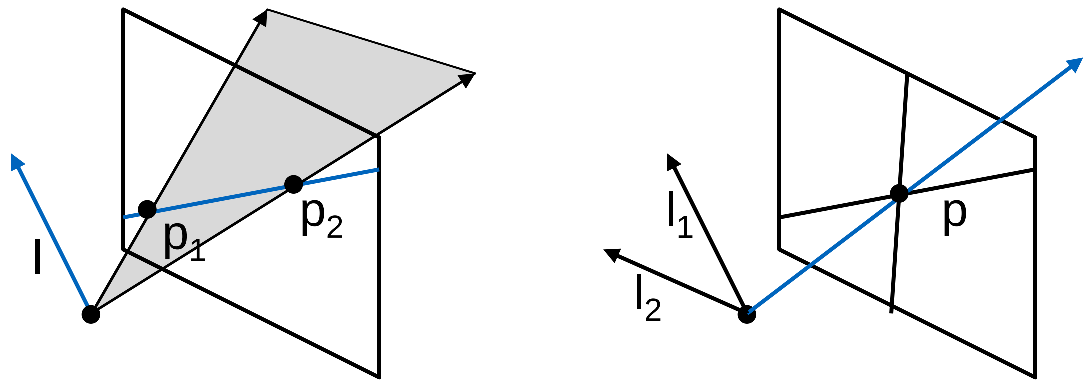
- To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

$$\mathbf{x} \longleftrightarrow \mathbf{l}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = 0 \longleftrightarrow \mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' \longleftrightarrow \mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

Point and line duality



What is the line **l** spanned by rays $\mathbf{p_1}$ and $\mathbf{p_2}$?

- **l** is $\perp$ to $\mathbf{p_1}$ and $\mathbf{p_2}$ $\Rightarrow$ $\mathbf{l} = \mathbf{p_1} \times \mathbf{p_2}$
- **l** is the plane normal

What is the intersection of two lines $\mathbf{l_1}$ and $\mathbf{l_2}$?

- **p** is $\perp$ to $\mathbf{l_1}$ and $\mathbf{l_2}$ $\Rightarrow$ $\mathbf{p} = \mathbf{l_1} \times \mathbf{l_2}$

Points and lines are dual in projective space

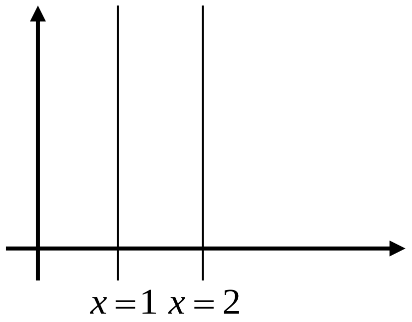
- given any formula, can switch the meanings of points and lines to get another formula

Intersection of parallel lines

- l and m are two parallel lines

$l = (a, b, c)^T$ $e.g. (-1,0,1)^T$ (a line parallel to y-axis)

$m = (a, b, d)^T$ $e.g. (-1,0,2)^T$ (another line parallel to y-axis)



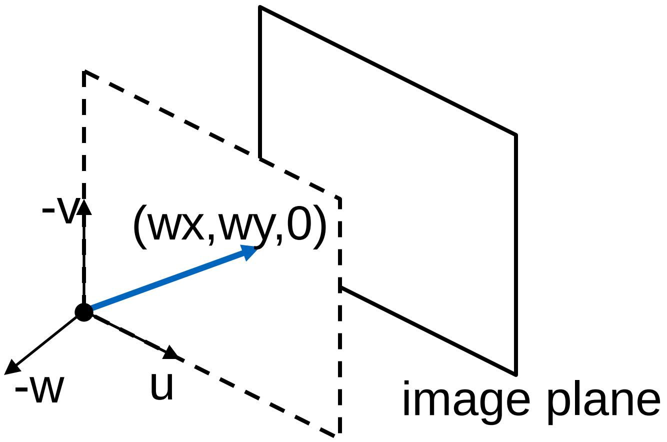
- Intersection of l and m

$$x = l \times m$$

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} a \\ b \\ d \end{bmatrix} = \begin{bmatrix} bd - bc \\ ac - ad \\ ab - ab \end{bmatrix} = (d - c) \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$

- A point (x,y,0) is called an ideal point, it does not lie in the image plane. But where does it lie then

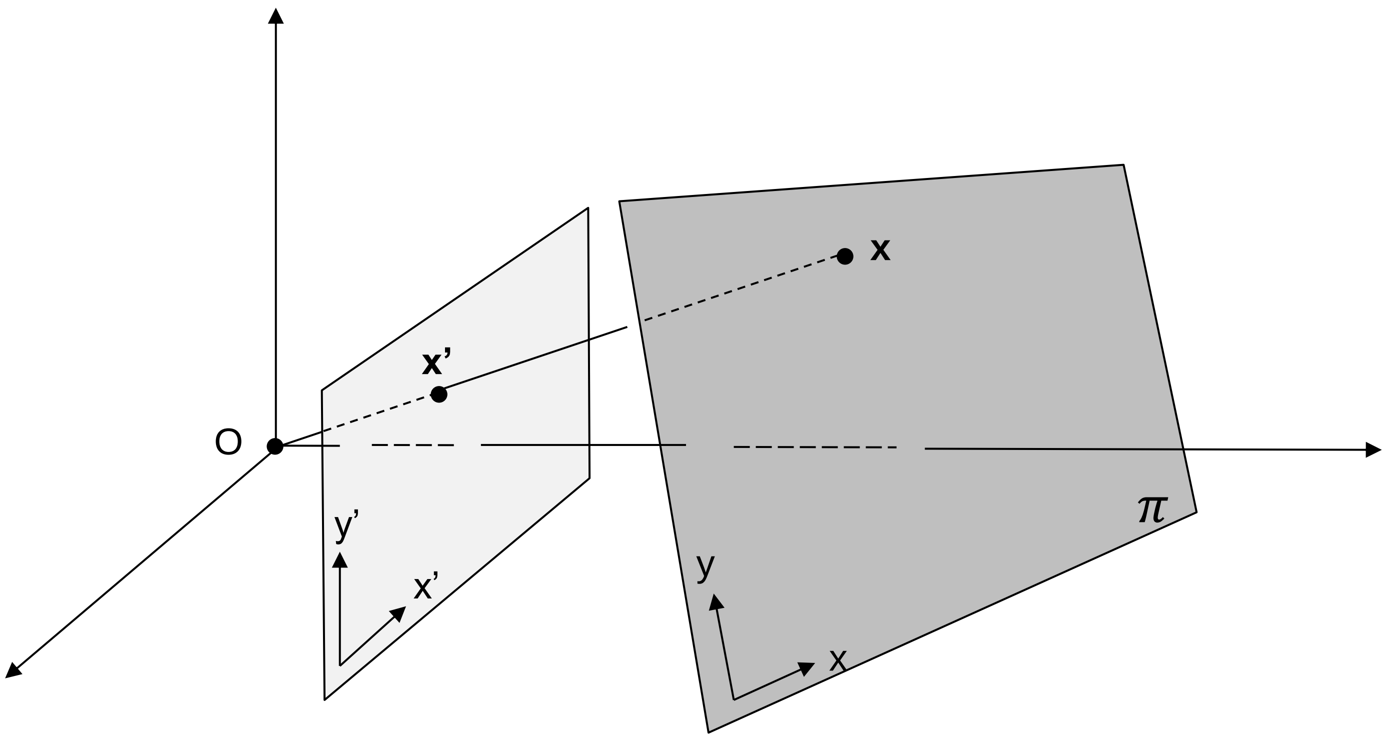
Ideal points and line at infinity



- Ideal point ("point at infinity")
  - p ≅ (x, y, 0) – parallel to image plane
  - It has infinite image coordinates
- All ideal points lie at the line at infinity
  - l ≅ (0, 0, 1) – normal to the image plane
  - Why is it called a line at infinity?

Projective transformations

- Mapping between planes x’=Hx



Projective transformations

Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or $\mathrm{x}' = \mathbf{H}\,\mathrm{x}$

8DOF

projectivity=collineation=projective transformation=homography

To transform a point: **p' = Hp**

To transform a line: **lp**=0 $\rightarrow$ **l'p'**=0

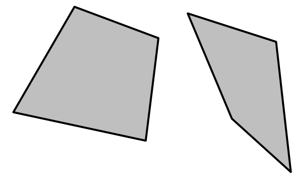
0 = **lp** = $\mathbf{l}\mathbf{H}^{-1}\mathbf{H}\mathbf{p}$ = $\mathbf{l}\mathbf{H}^{-1}\mathbf{p}'$ $\Rightarrow$ $\mathbf{l}' = \mathbf{l}\mathbf{H}^{-1}$

lines are transformed by postmultiplication of $\mathbf{H}^{-1}$

Overview 2D transformations

Projective
8dof

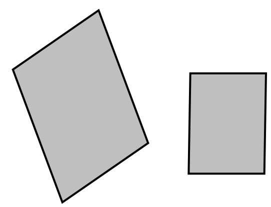
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio

Affine
6dof

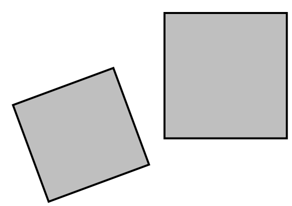
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).

Similarity
4dof

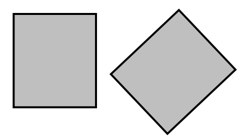
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.

Euclidean
3dof

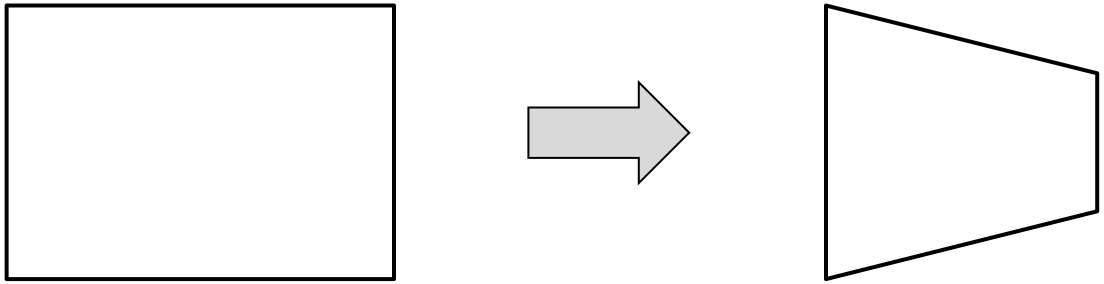
$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



lengths, areas.

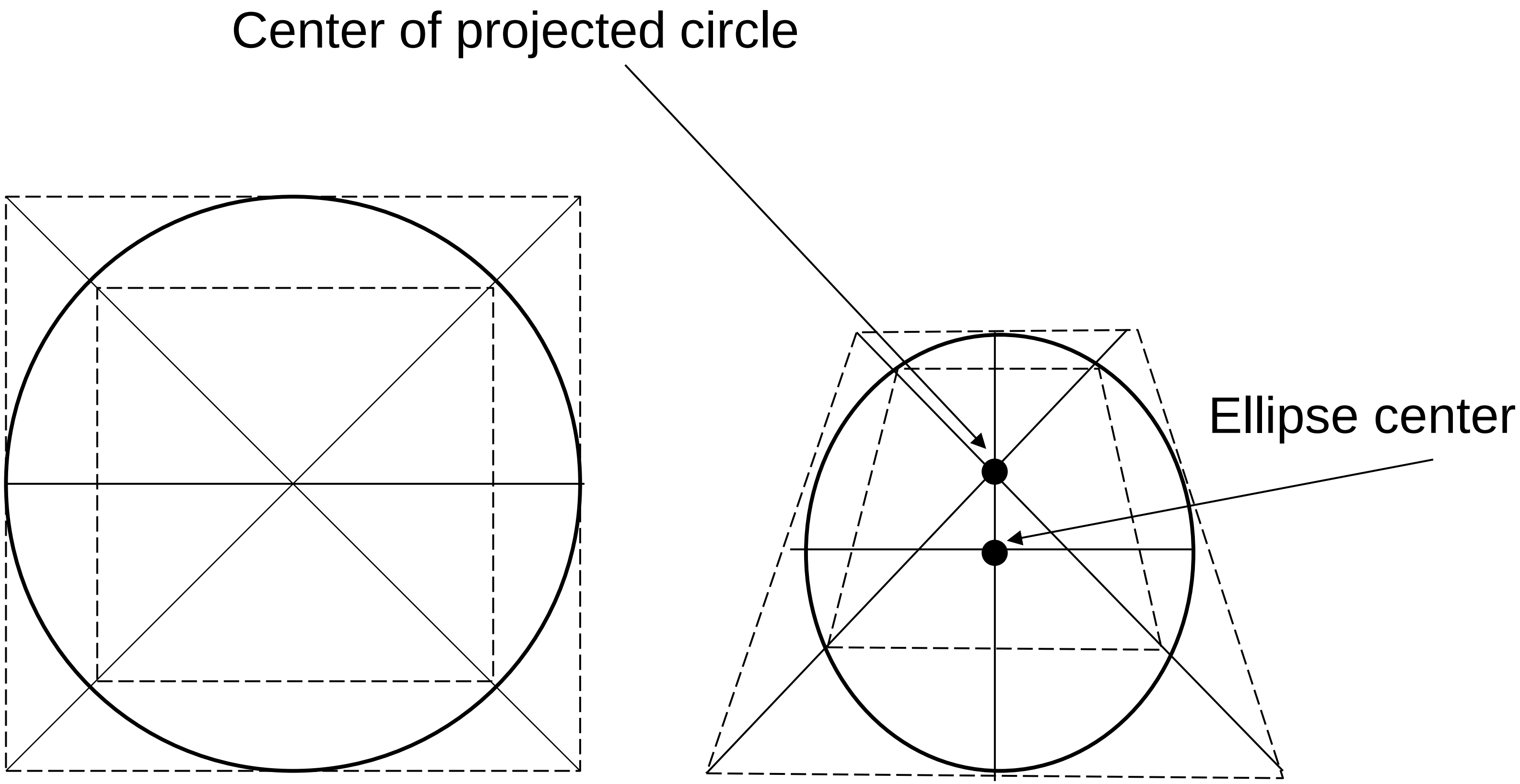
Effects of projective transformations

- Foreshortening effects can be imaged easily with primitive shapes



- But, how does a circle get transformed?

Effects of projective transformations



2D circle

Circle after projective transformation

Projective Geometry
3D Projective Geometry

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3D projective geometry

- Points, Lines, Planes
- Duality
- Plane at infinity

3D projective geometry

- The concepts of 2D generalize naturally to 3D
  - The axioms of geometry can be applied to 3D as well
- 3D projective space = 3D Euclidean space + plane at infinity
  - Not so simple to visualize anymore (4D space)
- Entities are now points, lines and planes
  - Projective 3D points have four coordinates: **P** = (x,y,z,w)
- Points, lines, and planes lead to more intersection and joining options than in the 2D case

Planes

- Plane equation

$$\Pi_1 X + \Pi_2 Y + \Pi_3 Z + \Pi_4 = 0$$
$$\Pi^T X = 0$$

  - Expresses that point X is on plane Π

- Plane parameters

$$\Pi = [\Pi_1, \Pi_2, \Pi_3, \Pi_4]$$

  - Plane parameters are normal vector + distance from origin

Join and incidence relations with planes

- A plane is defined uniquely by the join of three points, or the join of a line and point in general position
- Two distinct planes intersect in a unique line
- Three distinct planes intersect in a unique point

Three points define a plane

- X1,X2,X3 are three distinct points, each has to fullfil the incidence equation. Equations can be stacked.

$$\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix} \Pi = 0 \; (3x4)(4x1)$$

  - Plane parameters are the solution vector to this linear equation system (e.g. SVD)

- Points and planes are dual

$$\begin{bmatrix} \Pi_1^T \\ \Pi_2^T \\ \Pi_3^T \end{bmatrix} X = 0$$

Lines

- Lines are complicated
- Lines and points are not dual in 3D projective space
- Lines are represented by a 4x4 matrix, called Plücker matrix
- Computation of the line matrix from two points A,B

$$L = AB^T - BA^T \; (4x4) \; matrix$$

- Matrix is skew-symmetric
- Example line of the x-axis
  - $x1 = [0\ 0\ 0\ 1]^T$
    $x2 = [1\ 0\ 0\ 1]^T$
    $L = x1*x2^T - x2*x1^T$

$$L = \begin{matrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$$

Lines

- Points and planes are dual, we can get new equations by substituting points with planes

$$L = AB^T - BA^T \ (A, B \ are \ points)$$
$$L^* = PQ^T - QP^T \ (P, Q \ are \ planes)$$

- The intersection of two planes P,Q is a line
- Lines are self dual, the same line L has a dual representation L*
- The matrix L can be directly computed from the entries of L*

$$\begin{aligned}
\ell_{12} &= \ell^*_{34} \\
\ell_{13} &= \ell^*_{42} \\
\ell_{14} &= \ell^*_{23} \\
\ell_{23} &= \ell^*_{14} \\
\ell_{42} &= \ell^*_{13} \\
\ell_{34} &= \ell^*_{12}
\end{aligned}$$

Point, planes and lines

- A plane can be defined by the join of a point X and a line L

$$\Pi = L^{*} X$$

- A point can be defined by the intersection of a plane with a line L

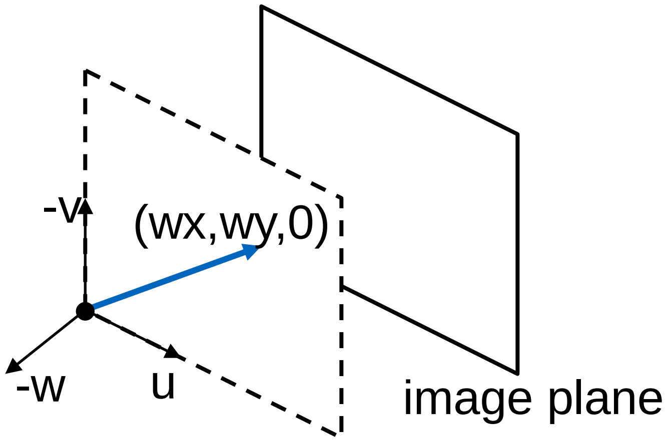
$$X = L\Pi$$

Plane at infinity

- Parallel lines and parallel planes intersect at Π_∞
- Plane parameters of Π_∞

$$\Pi_\infty = (0,0,0,1)^T$$

- It is a plane that contains all the direction vectors $D = (x1, x2, x3, 0)^T$, vectors that originate from the origin of 4D space
- Try to imagine an extension of the 2D case (see illustration below) to the 3D case…



Recap - Learning goals

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